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Simple Soliton Solution Method for the Combined KdV and MKdV Equation

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Malfliet first proposed a simple solution method for the multisoliton solution of the KdV equation. Abdel-Rahman used Malfliet's method in a slightly modified form, and gave the multisoliton solution of the mKdV equation, RLW equation, Boussinesq equation, and modified Boussinesq equation. In this paper, we solve the soliton solution of the cKdV–mKdV equation by using this method.

1. INTRODUCTION

An interesting method for the multisoliton solution of the KdV equation was proposed by Malfliet [1]. The method is much simpler than the intricate inverse scattering transform approach of Gardner *et al.* [2], which applies to a variety of nonlinear evolution equations. Abdel-Rahman [3] further obtained the multisoliton solution of the mKdV equation, the RLW equation, the Boussinesq equation, and the modified Boussinesq equation by using Malfliet's method in a slightly modified form. In this paper we develop this method and obtain the multisoliton solution of the combined KdV and mKdV equation (cKdV–mKdV E)

$$u_t + 6uu_x + 6u^2u_x + u_{xxx} = 0 \tag{1}$$

Equation (1) is widely used in such fields as solid-states physics, plasma physics, fluid physics, and quantum field theory [4, 5]. Equation (1) is integrable, which means that it has a Bäcklund transformation [5], has a bilinear form, a Lax pair, and an infinite number of conservation laws [6] Coffey [7] solved Eq. (1) by a series expansion method [8]. Mohamad [9]

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solved the traveling wave solutions by both a direct method and a leading order analysis method. Lou and Chen [10] presented a mapping approach for the construction of exact solitary wave solutions and cnoidal wave solutions of Eq. (1).

2. SOLUTION METHOD

Performing a dependent variable transformation $u \to u/\lambda$, $t \to \lambda^2 t$, $x \to \lambda x$, where λ is as yet an arbitrary parameter, we get

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{6}{\lambda}u^2\frac{\partial u}{\partial x} + \frac{1}{\lambda}\frac{\partial^3 u}{\partial x^3} = 0 \tag{2}$$

Next we change the dependent variables (x, t) to (ξ, τ), where $\xi = x - vt$ (v an arbitrary parameter) and $\tau = t$, and require u to depend on ξ only, i.e., $u = u(\xi)$, $\partial u/\partial\tau = 0$. This implies that in a frame traveling with velocity v, we see a stationary wave u. Equation (2) then becomes

$$-v\frac{du}{d\xi} + 6u\frac{du}{d\xi} + \frac{6}{\lambda}u^2\frac{du}{d\xi} + \frac{1}{\lambda}\frac{d^3u}{d\xi^3} = 0 \tag{3}$$

Following the method of refs. 1 and 3, we write

$$u(\xi) = f(\xi)g(\xi) \tag{4}$$

where $f(\xi)$ and $g(\xi)$ are two arbitrary functions. Then

$$\frac{du}{d\xi} = \left(f\frac{dg}{d\xi}\right) + (f \leftrightarrow g) \tag{5}$$

$$\frac{d^3u}{d\xi^3} = \left(f\frac{d^3g}{d\xi^3} + 3\frac{d^2f}{d\xi^2}\frac{dg}{d\xi}\right) + (f \leftrightarrow g) \tag{6}$$

and we write

$$\begin{aligned} 6u\frac{du}{d\xi} &= n_1 u\frac{du}{d\xi} + mu\frac{du}{d\xi} \\ &= n_1 fg\frac{du}{d\xi} + mu\left(f\frac{dg}{d\xi} + g\frac{df}{d\xi}\right) \end{aligned} \tag{7}$$

$$\frac{6}{\lambda}u^2\frac{du}{d\xi} = n_2 u^2\frac{du}{d\xi} + mu^2\frac{du}{d\xi}$$

$$= n_2 fgu \frac{du}{d\xi} + mu^2\left(f \frac{dg}{d\xi} + g \frac{df}{d\xi}\right) \tag{8}$$

where $n_1 + m = 6$, $n_2 + m = 6/\lambda$.

Substitution of Eqs. (5)–(8) into Eq. (3) then gives

$$f\left[\frac{1}{\lambda} \frac{d^3 g}{d\xi^3} + \frac{3}{\lambda} \frac{1}{f} \frac{d^2 f}{d\xi^2} \frac{dg}{d\xi} + m\left(u^2 + u - \frac{v}{m}\right) \frac{dg}{d\xi} + \frac{1}{2} g(n_1 + n_2 u) \frac{du}{d\xi}\right] + g[f \leftrightarrow g] = 0 \tag{9}$$

Motivated by the occurrence of $dg/d\xi$ in both the second and third terms, we utilize the arbitrariness in f and g and set

$$\frac{1}{\lambda} \frac{1}{f} \frac{d^2 f}{d\xi^2} = -\left(u^2 + u - \frac{v}{m}\right) \tag{10}$$

with a similar equation for g. Thus with $\psi = f, g$ we have the Schrödinger equation

$$\frac{1}{\lambda} \frac{d^2 \psi}{d\xi^2} + \left(u^2 + u - \frac{v}{m}\right)\psi = 0 \tag{11}$$

in which $u^2 + u$ is the scattering potential and v/m the eigenvalue. Substitution of Eq. (11) ($\psi = f, g$) into Eq. (9) gives

$$f\left[\frac{1}{\lambda} \frac{d^3 g}{d\xi^3} - (3 - m)\left(u^2 + u - \frac{v}{m}\right) \frac{dg}{d\xi} + \frac{1}{2} g(n_1 + n_2 u) \frac{du}{d\xi}\right] + g[f \leftrightarrow g] = 0 \tag{12}$$

On the other hand, differentiating Eq. (11) with respect to ξ, we get

$$\frac{1}{\lambda} \frac{d^3 \psi}{d\xi^3} + \left(u^2 + u - \frac{v}{m}\right) \frac{d\psi}{d\xi} + \psi(2u + 1) \frac{du}{d\xi} = 0 \tag{13}$$

which coincides with Eq. (12) provided $m = 4$, $n_1 = 2$, $n_2 = 4$, i.e., $6/\lambda = 8$ or $\lambda = 3/4$. Thus each of the two functions f and g satisfies the same Schrödinger equation

$$\psi_{\xi\xi} + \frac{3}{4}\left(u^2 + u - \frac{v}{4}\right)\psi = 0 \tag{14}$$

We now assume that the potential is $-(u^2 + u) < 0$ and N distinct discrete eigenvalues $-k_n^2 = -v_n/4$ $(n = 1, 2, \ldots, N)$ associated with it.

So we rewrite Eq. (14) as

$$\frac{d^2\psi_n}{d\xi^2} + \frac{3}{4}(u^2 + u - k_n^2)\psi_n = 0, \qquad \xi_n = x - 4k_n^2 t \tag{15}$$

Considering (4), the general solution of Eq. (1) can be written as

$$u = \psi_n^2(\xi_n) \tag{16}$$

If the wave functions ψ_n do not overlap with each other, we may write Eq. (15) as

$$\frac{d^2\psi_n}{d\xi^2} + \frac{3}{4}(\psi_n^4 + \psi_n^2 - k_n^2)\psi_n = 0 \tag{17}$$

i.e.,

$$\frac{d^2\psi_n}{d\xi^2} - \frac{3}{4}k_n^2\psi_n + \frac{3}{4}\psi_n^3 + \frac{3}{4}\psi_n^5 = 0 \tag{18}$$

In order to solve Eq. (18), let

$$\psi_n(\xi) = \sqrt{\varphi_n(\xi)} \tag{19}$$

then by Eq. (18) we get the following equation satisfied by $\varphi_n(\xi)$:

$$2\varphi\frac{d^2\varphi_n}{d\xi_n} - \left(\frac{d\varphi_n}{d\xi}\right)^2 - 3k_n^2\varphi_n^2 + 3\varphi_n^3 + 3\varphi_n^4 = 0 \tag{20}$$

We suggest that (20) might find a solution of the following form:

$$\varphi_n(\xi) = \frac{A\exp[\alpha(\xi + \xi_0)]}{\{1 + \exp[\alpha(\xi + \xi_0)]\}^2 + B\exp[\alpha(\xi + \xi_0)]} \tag{21}$$

where A, B, α are constants to be determined, and ξ_0 is a fixed real number.

By (21), we get

$$\frac{d\varphi_n}{d\xi} = \frac{A\,\alpha\exp[\alpha(\xi + \xi_0)] - A\,\alpha\exp[3\alpha(\xi + \xi_0)]}{\{1 + \exp[\alpha(\xi + \xi_0)]\}^2 + B\exp[\alpha(\xi + \xi_0)]} \tag{22}$$

and

$$\frac{d^2\varphi_n}{d\xi^2} = \frac{A\,\alpha^2\exp[\alpha(\xi + \xi_0)]\phi(\xi)}{(\{1 + \exp[\alpha(\xi + \xi_0)]\}^2 + B\exp[\alpha(\xi + \xi_0)])^3} \tag{23}$$

where

$$\phi(\xi) = 1 - (2 + B)\exp[\alpha(\xi + \xi_0)] - 6\exp[2\alpha(\xi + \xi_0)] - (2 + B)\exp[3\alpha(\xi + \xi_0)] + \exp[4\alpha(\xi + \xi_0)] \tag{24}$$

Substituting (21)–(24) into (20), we get the following equations satisfied by A, B, and α:

$$\alpha^2 - 3k_n^2 = 0 \tag{25}$$

$$\frac{3}{2}A - 3k_n^2(2 + B) - \alpha^2(2 + B) = 0 \tag{26}$$

$$-5\alpha^2 - \frac{3}{2}k_n^2[2 + (2 + B)^2] + \frac{3}{2}A(2 + B) + \frac{3}{2}A^2 = 0 \tag{27}$$

It follows from (25)–(27) that

$$\alpha = \sqrt{3}k_n \tag{28}$$

$$A = \pm\frac{16k_n}{\sqrt{3 + 16k_n^2}} \tag{29}$$

$$B = -2 \pm 2\sqrt{\frac{3}{3 + 16k_n^2}} \tag{30}$$

or

$$\alpha = -\sqrt{3}k_n \tag{31}$$

$$A = \pm\frac{16k_n}{\sqrt{3 + 16k_n^2}} \tag{32}$$

$$B = -2 \pm 2\sqrt{\frac{3}{3 + 16k_n^2}} \tag{33}$$

Substituting (28)–(33) into Eq. (20), we have the following solutions of Eq. (20):

$$\varphi_{n1}(\xi) = \frac{16k_n}{\sqrt{3 + 16k_n^2}}\exp[\sqrt{3}k_n(\xi + \xi_0)] \times \left(\{1 + \exp[\sqrt{3}k_n(\xi + \xi_0)]\}^2 + \left(-2 + 2\sqrt{\frac{3}{3 + 16k_n^2}}\right)\exp[\sqrt{3}k_n(\xi + \xi_0)]\right)^{-1} \tag{34}$$

$$\varphi_{n2}(\xi) = -\frac{16k_n}{\sqrt{3+16k_n{}^2}}\exp[\sqrt{3}k_n(\xi+\xi_0)]$$
$$\times(\{1+\exp[\sqrt{3}k_n(\xi+\xi_0)]\}^2$$
$$+\left(-2-2\sqrt{\frac{3}{3+16k_n{}^2}}\right)\exp[\sqrt{3}k_n(\xi+\xi_0)])^{-1} \tag{35}$$

Solutions (34) and (35) can be transformed into

$$\varphi_n(\xi) = \pm\frac{[8k_n/\sqrt{(3+16k_n{}^2)}]\,\mathrm{sech}^2[(\sqrt{3}/2)k_n(\xi+\xi_0)]}{2+[-1+\sqrt{3/(3+16k_n{}^2)}]\,\mathrm{sech}^2[(\sqrt{3}/2)k_n(\xi+\xi_0)]} \tag{36}$$

Thus from Eqs. (19) and (16), Eq. (1) admits the following exact soliton solution:

$$u = \psi_n^2 = \frac{[8k_n/\sqrt{(3+16k_n{}^2)}]\,\mathrm{sech}^2[(\sqrt{3}/2)k_n(\xi+\xi_0)]}{2+[-1+\sqrt{3/(3+16k_n{}^2)}]\,\mathrm{sech}^2[(\sqrt{3}/2)k_n(\xi+\xi_0)]} \tag{37}$$

3. CONCLUSION

We have used a simple solution method introduced by Malfliet and generalized by Abdel-Rahman in order to obtain the soliton solutions for the cKdV–mKdV equation (1). Further applications of this method are under study.

REFERENCES

1. W. Malfliet (1981). *Am. J. Phys.* **49**, 666.
2. C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura (1974). *Commun. Pure Appl. Math.* **27**, 97.
3. A.-M. M. Abdel-Rahman (1983). *Am. J. Phys.* **51**, 510.
4. B. Dey (1986). *J. Phys. A Math. Gen.* **19**, L9; K. Konno and Y. H. Ichikawa (1974). *J. Phys. Soc. Jpn.* **37**, 1631; V. Narayanamurit and C. M. Varma (1970). *Phys. Rev. Lett.* **25**, 1105; F. D. Tappert and C. M. Varma (1990). *Phys, Rev. Lett* **25**, 1108.
5. M. Wadati (1975). *J. Phys. Soc. Jpn.* **38**, 673, 681.
6. R. M. Miura (1968). *J. Math. Phys.* **9**, 1202, 1204.
7. M. W. Coffey (1990). *SIAM J. Appl. Math.* **50**, 1580.
8. W. Hereman *et al.* (1986). *J. Phys. A Math. Gen.* **19**, 607.
9. M. N. B. Mohamad (1992). *Math. Meth. Appl. Sci.* **15**, 73.
10. S. Y. Lou and L. L. Chen (1994). *Math. Meth. Appl. Sci.* **17**, 339.